

**GELFAND-KIRILLOV CONJECTURE
AND GELFAND-TSETLIN MODULES FOR
FINITE W -ALGEBRAS**

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ABSTRACT. We address two problems regarding the structure and representation theory of finite W -algebras associated with the general linear Lie algebras. Finite W -algebras can be defined either via the Whittaker modules of Kostant or, equivalently, by the quantum Hamiltonian reduction. Our first main result is a proof of the Gelfand-Kirillov conjecture for the skew fields of fractions of the finite W -algebras. The second main result is a parametrization of finite families of irreducible Gelfand-Tsetlin modules by the characters of the Gelfand-Tsetlin subalgebra. As a corollary, we obtain a complete classification of generic irreducible Gelfand-Tsetlin modules for the finite W -algebras.

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1. INTRODUCTION

The concept of finite W -algebras goes back to the original paper of Kostant [Ko] dealing with the study of Whittaker modules and to its generalization by Lynch [L]. An alternative construction of W -algebras can be given via the quantum Hamiltonian reduction which goes back to the works of Feigin and Frenkel [FF], Kac, Roan and Wakimoto [KRW], Kac and Wakimoto [KW] and De Sole and Kac [SK]. It was shown by D'Andrea, De Concini, De Sole, Heluani and Kac [SK, Appendix] and by Arakawa [A] that both definitions of finite W -algebras are equivalent.

Let $\mathfrak{g} = \mathfrak{gl}_m$ denote the general linear Lie algebra over an algebraically closed field \mathbb{k} of characteristic 0 which will be fixed throughout the paper. A finite W -algebra can be associated to a fixed nilpotent element $f \in \mathfrak{g}$ as follows. A \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is called a *good grading* for f if $f \in \mathfrak{g}_2$ and the linear map

$$\mathrm{ad} f : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$$

is injective for $j \leq -1$ and surjective for $j \geq -1$. A complete classification of good gradings for simple Lie algebras was given by Elashvili and Kac [EK]. A non-degenerate invariant symmetric bilinear form (\cdot, \cdot) on \mathfrak{g} induces a non-degenerate skew-symmetric form on \mathfrak{g}_{-1} defined by $\langle x, y \rangle = ([x, y], f)$. Let $\mathcal{I} \subset \mathfrak{g}_{-1}$ be a maximal isotropic subspace and set $\mathfrak{t} = \bigoplus_{j \leq -2} \mathfrak{g}_j \oplus \mathcal{I}$. Now let $\chi : U(\mathfrak{t}) \rightarrow \mathbb{C}$ be the one-dimensional representation such that $x \mapsto (x, f)$ for any $x \in \mathfrak{t}$. Set $I_\chi = \mathrm{Ker} \chi$ and $Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})I_\chi$. The corresponding finite W -algebra is defined by

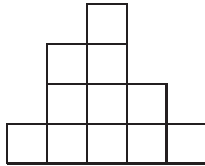
$$W(\chi) = \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

If the grading on \mathfrak{g} is *even*, i.e. $\mathfrak{g}_j = 0$ for all odd j , then $W(\chi)$ is isomorphic to the subalgebra of \mathfrak{t} -twisted invariants in $U(\mathfrak{p})$ for the parabolic subalgebra $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. Note that by the results of Elashvili and Kac [EK], it is sufficient to consider only even good gradings.

The growing interest to the theory of finite W -algebras is due, on the one hand, to their geometric realizations as quantizations of the Slodowy slices (see Premet [P] and Gan and Ginzburg [GG]), and, on the other hand, to their close connections with the Yangian theory which was originally observed by Ragoucy and Sorba [RS] and developed in full generality by Brundan and Kleshchev [BK1]. The latter results may well be regarded as a substantial step forward in understanding the structure of the finite W -algebras associated to \mathfrak{gl}_m . These algebras turn out to be isomorphic to certain quotients of the *shifted Yangians*, which provides their presentations in terms of generators and defining relations and thus opens the way for developing the representation theory for the finite W -algebras; see [BK2].

In more detail, following [EK], consider a *pyramid* π which is a unimodal sequence (q_1, q_2, \dots, q_l) of positive integers with $q_1 \leq \dots \leq q_k$ and $q_{k+1} \geq \dots \geq q_l$ for some $0 \leq k \leq l$. Such a pyramid can be visualized as the diagram of bricks (unit squares)

which consists of q_1 bricks stacked in the first (leftmost) column, q_2 bricks stacked in the second column, etc. The pyramid π defines the tuple (p_1, \dots, p_n) of its row lengths, where p_i is the number of bricks in the i th row of the pyramid, so that $1 \leq p_1 \leq \dots \leq p_n$. The figure illustrates the pyramid with the columns $(1, 3, 4, 2, 1)$ and rows $(1, 2, 3, 5)$:



If the total number of bricks in the pyramid π is m , then the finite W -algebra $W(\pi)$ associated to \mathfrak{gl}_m corresponds to the nilpotent matrix $f \in \mathfrak{gl}_m$ of Jordan type (p_1, \dots, p_n) ; see Section 2 for the precise definition and the relationship of $W(\pi)$ with the shifted Yangian. One of surprising consequences of the results of [BK1] is that the isomorphism class of $W(\pi)$ depends only on the sequence of row lengths (p_1, \dots, p_n) of π . Therefore, we may assume without restricting generality that the rows of π are left-justified.

The first problem we address in this paper is the *Gelfand-Kirillov conjecture* for the algebras $W(\pi)$. The original conjecture states that the universal enveloping algebra of an algebraic Lie algebra over an algebraically closed field is "birationally" equivalent to some Weyl algebra over a purely transcendental extension of \mathbb{k} , i.e. its skew field of fractions is a Weyl field. The conjecture was settled in the original paper by Gelfand and Kirillov [GK1] for nilpotent Lie algebras, and for \mathfrak{gl}_m and \mathfrak{sl}_m ; see also [GK2], where its weaker form was proved. For solvable Lie algebras the conjecture was settled by Borho, Gabriel and Rentschler [BGR], Joseph [Jo] and McConnell [Mc]. Some mixed cases were considered by Nghiem [Ng], while Alev, Ooms and Van den Bergh [AOV1] proved the conjecture for all Lie algebras of dimension at most eight. On the other hand, counterexamples to the conjecture are known for certain semi-direct products; see e.g. [AOV2]. We refer the reader to the book by Brown and Goodearl [BG] and references therein for generalizations of the Gelfand-Kirillov conjecture for quantized enveloping algebras.

For an associative algebra A we denote by $D(A)$ its skew field of fractions, if it exists. Let A_k be the k -th Weyl algebra over \mathbb{k} and $D_k = D(A_k)$ its skew field of fractions. Let \mathcal{F} be a pure transcendental extension of \mathbb{k} of degree m and let $A_k(\mathcal{F})$ be the k -th Weyl algebra over \mathcal{F} . Denote by $D_{k,m}$ the skew field of fractions of $A_k(\mathcal{F})$.

Gelfand-Kirillov problem for $W(\pi)$: Does $D(W(\pi)) \simeq D_{k,m}$ for some k, m ?

Our first main result is a positive solution of this problem.

Theorem I. The Gelfand-Kirillov conjecture holds for $W(\pi)$:

$$D(W(\pi)) \simeq D_{k,m},$$

where $k = \sum_{i=1}^l q_i(q_i - 1)/2$ and $m = q_1 + \dots + q_l$.

Note that m is the number of bricks in the pyramid π , while k can be interpreted as the sum of all leg lengths of the bricks. Hence, k and m can be expressed in terms of the rows as $k = (n-1)p_1 + \dots + p_{n-1}$ and $m = p_1 + \dots + p_n$. In the case of the one-column pyramid $(1, \dots, 1)$ of height m we recover the original result of [GK1] for \mathfrak{gl}_m . One of the key points in the proof of Theorem I is a positive solution of the *noncommutative Noether problem* for the symmetric group S_k :

Noncommutative Noether problem for S_k : Does $D_k^{S_k} \simeq D_k$?

Here S_k acts naturally on A_k and on D_k by simultaneous permutations of variables and derivations.

The second problem that we address in this paper is the classification problem of irreducible Gelfand-Tsetlin modules (sometimes also called Harish-Chandra modules) for finite W -algebras with respect to the Gelfand-Tsetlin subalgebra. Given a pyramid π with the left-justified rows (p_1, \dots, p_n) , for each $k \in \{1, \dots, n\}$ we let π_k denote the pyramid (p_1, \dots, p_k) . We have the chain of natural subalgebras

$$(1.1) \quad W(\pi_1) \subset W(\pi_2) \subset \dots \subset W(\pi_n) = W(\pi).$$

Denote by Γ the (commutative) subalgebra of $W(\pi)$ generated by the centers of the subalgebras $W(\pi_k)$ for $k = 1, \dots, n$. Note that the structure of the center of the algebra $W(\pi)$ is described in [BK2, Theorem 6.10]. Following the terminology of that paper, we call Γ the *Gelfand-Tsetlin subalgebra* of $W(\pi)$.

A finitely generated module M over $W(\pi)$ is called a *Gelfand-Tsetlin module* (with respect to Γ) if

$$M = \bigoplus_{\mathbf{m} \in \text{Specm } \Gamma} M(\mathbf{m})$$

as a Γ -module, where

$$M(\mathbf{m}) = \{x \in M \mid \mathbf{m}^k x = 0 \text{ for some } k \geq 0\}$$

and $\text{Specm } \Gamma$ denotes the set of maximal ideals of Γ . In the case of the one-column pyramids π this reduces to the definition of the Gelfand-Tsetlin modules for \mathfrak{gl}_m [DFO1]. Note also that the *admissible* $W(\pi)$ -modules of [BK2] are Gelfand-Tsetlin modules.

An irreducible Gelfand-Tsetlin module M is said to be *extended* from $\mathbf{m} \in \text{Specm } \Gamma$ if $M(\mathbf{m}) \neq 0$. The set of isomorphism classes of irreducible Gelfand-Tsetlin modules extended from \mathbf{m} is called the *fiber* of $\mathbf{m} \in \text{Specm } \Gamma$. Equivalently, this is the set of left maximal ideals of $W(\pi)$ containing \mathbf{m} . An important problem in the theory of Gelfand-Tsetlin modules is to determine the cardinality of the fiber of an arbitrary \mathbf{m} . In the case where the fibers consist of single isomorphism classes, the corresponding irreducible Gelfand-Tsetlin modules are parameterized by the elements of $\text{Specm } \Gamma$.

This problem was solved in the particular cases of one-column pyramids [O] (\mathfrak{gl}_n case) and two-row rectangular pyramids [FMO1] (Yangian for \mathfrak{gl}_2). We extend these results to arbitrary finite W -algebras of type A . The technique used in this paper is quite different, it is based on the properties of the *Galois orders* developed in the papers [FO1] and [FO2]. Our second main result is the following theorem.

Theorem II. *The fiber of any $\mathbf{m} \in \text{Specm } \Gamma$ in the category of Gelfand-Tsetlin modules over $W(\pi)$ is non-empty and finite.*

Clearly, the same irreducible Gelfand-Tsetlin module can be extended from different maximal ideals of Γ ; such ideals are called *equivalent*. Hence, Theorem II provides a parametrization of finite families of irreducible Gelfand-Tsetlin modules over $W(\pi)$ by the equivalence classes of characters of the Gelfand-Tsetlin subalgebra. Moreover, this gives a classification of the irreducible *generic* Gelfand-Tsetlin modules. In order to formulate the result, recall that a non-empty set $X \subset \text{Specm } \Gamma$ is called *massive* if X contains the intersection of countably many dense open subsets. If the field \mathbb{k} is uncountable, then a massive set X is dense in $\text{Specm } \Gamma$.

Theorem III. *There exists a massive subset $\Omega \subset \text{Specm } \Gamma$ such that*

- (i) *For any $\mathbf{m} \in \Omega$, there exists a unique, up to isomorphism, irreducible module $L_{\mathbf{m}}$ over $W(\pi)$ in the fiber of \mathbf{m} .*
- (ii) *For any $\mathbf{m} \in \Omega$ the extension category generated by $L_{\mathbf{m}}$ contains all indecomposable modules whose support contains \mathbf{m} and is equivalent to the category of modules over the algebra of formal power series in $n p_1 + (n - 1) p_2 + \dots + p_n$ variables.*

We also make the following conjecture about the size of fiber in general:

Conjecture 1. Let (p_1, \dots, p_n) be the rows of π . For any $\mathbf{m} \in \text{Specm } \Gamma$ the fiber of \mathbf{m} consists of at most $p_1!(p_1 + p_2)! \dots (p_1 + \dots + p_{n-1})!$ isomorphism classes of irreducible Gelfand-Tsetlin $W(\pi)$ -modules. The same bound holds for the dimension of the subspace of \mathbf{m} -nilpotents $V(\mathbf{m})$ in any irreducible Gelfand-Tsetlin module V .

This conjecture follows immediately from Theorem 5.3(iii) in [FO2] and the following conjecture.

Conjecture 2. $W(\pi)$ is free as left (right) module over the Gelfand-Tsetlin subalgebra.

These conjectures known to be true in the particular cases of one-column pyramids [O] and two-row rectangular pyramids [FMO1]. We prove both conjectures for arbitrary two-row pyramids (i.e., finite W -algebras associated with \mathfrak{gl}_2).

2. SHIFTED YANGIANS, FINITE W -ALGEBRAS AND THEIR REPRESENTATIONS

As in [BK1], given a pyramid π with the rows $p_1 \leq \dots \leq p_n$, introduce the corresponding *shifted Yangian* $Y_\pi(\mathfrak{gl}_n)$ as the associative algebra over \mathbb{k} defined by generators

$$(2.1) \quad \begin{aligned} d_i^{(r)}, \quad i &= 1, \dots, n, & r &\geq 1, \\ f_i^{(r)}, \quad i &= 1, \dots, n-1, & r &\geq 1, \\ e_i^{(r)}, \quad i &= 1, \dots, n-1, & r &\geq p_{i+1} - p_i + 1, \end{aligned}$$

subject to the following relations:

$$\begin{aligned} [d_i^{(r)}, d_j^{(s)}] &= 0, \\ [e_i^{(r)}, f_j^{(s)}] &= -\delta_{ij} \sum_{t=0}^{r+s-1} d_i'^{(t)} d_{i+1}^{(r+s-t-1)}, \\ [d_i^{(r)}, e_j^{(s)}] &= (\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-t-1)}, \\ [d_i^{(r)}, f_j^{(s)}] &= (\delta_{i,j+1} - \delta_{ij}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)} d_i^{(t)}, \\ [e_i^{(r)}, e_i^{(s+1)}] - [e_i^{(r+1)}, e_i^{(s)}] &= e_i^{(r)} e_i^{(s)} + e_i^{(s)} e_i^{(r)}, \\ [f_i^{(r+1)}, f_i^{(s)}] - [f_i^{(r)}, f_i^{(s+1)}] &= f_i^{(r)} f_i^{(s)} + f_i^{(s)} f_i^{(r)}, \\ [e_i^{(r)}, e_{i+1}^{(s+1)}] - [e_i^{(r+1)}, e_{i+1}^{(s)}] &= -e_i^{(r)} e_{i+1}^{(s)}, \\ [f_i^{(r+1)}, f_{i+1}^{(s)}] - [f_i^{(r)}, f_{i+1}^{(s+1)}] &= -f_{i+1}^{(s)} f_i^{(r)}, \\ [e_i^{(r)}, e_j^{(s)}] &= 0 & \text{if } |i-j| > 1, \\ [f_i^{(r)}, f_j^{(s)}] &= 0 & \text{if } |i-j| > 1, \\ [e_i^{(r)}, [e_i^{(s)}, e_j^{(t)}]] + [e_i^{(s)}, [e_i^{(r)}, e_j^{(t)}]] &= 0 & \text{if } |i-j| = 1, \\ [f_i^{(r)}, [f_i^{(s)}, f_j^{(t)}]] + [f_i^{(s)}, [f_i^{(r)}, f_j^{(t)}]] &= 0 & \text{if } |i-j| = 1, \end{aligned}$$

for all admissible i, j, r, s, t , where $d_i^{(0)} = 1$ and the elements $d_i'^{(r)}$ are found from the relations

$$\sum_{t=0}^r d_i^{(t)} d_i'^{(r-t)} = \delta_{r0}, \quad r = 0, 1, \dots$$

Note that the algebra $Y_\pi(\mathfrak{gl}_n)$ depends only on the differences $p_{i+1} - p_i$ and our definition corresponds to the left-justified pyramid π , as compared to [BK1]. In the particular case of a rectangular pyramid π with $p_1 = \dots = p_n$, the algebra $Y_\pi(\mathfrak{gl}_n)$ is

isomorphic to the *Yangian* $Y(\mathfrak{gl}_n)$; see e.g. [M] for the description of its structure and representations. Moreover, for an arbitrary pyramid π , the shifted Yangian $Y_\pi(\mathfrak{gl}_n)$ can be regarded as a natural subalgebra of $Y(\mathfrak{gl}_n)$.

Due to the main result of [BK1], the *finite W -algebra* $W(\pi)$, associated to \mathfrak{gl}_m and the pyramid π , can be defined as the quotient of $Y_\pi(\mathfrak{gl}_n)$ by the two-sided ideal generated by all elements $d_1^{(r)}$ with $r \geq p_1 + 1$. We refer the reader to [BK1, BK2] for the description of the structure of the algebra $W(\pi)$, including analogues of the Poincaré–Birkhoff–Witt theorem and a construction of algebraically independent generators of the center.

2.1. Gelfand–Tsetlin basis for finite-dimensional representations. An important role in our arguments will be played by an explicit construction of a family of finite-dimensional irreducible representations of $W(\pi)$, given in [FMO2]. We reproduce some of the formulas here.

Introduce formal generating series in u^{-1} with coefficients in $W(\pi)$ by

$$\begin{aligned} d_i(u) &= 1 + \sum_{r=1}^{\infty} d_i^{(r)} u^{-r}, & f_i(u) &= \sum_{r=1}^{\infty} f_i^{(r)} u^{-r}, \\ e_i(u) &= \sum_{r=p_{i+1}-p_i+1}^{\infty} e_i^{(r)} u^{-r} \end{aligned}$$

and set

$$A_i(u) = u^{p_1} (u-1)^{p_2} \dots (u-i+1)^{p_i} a_i(u)$$

for $i = 1, \dots, n$ with $a_i(u) = d_1(u) d_2(u-1) \dots d_i(u-i+1)$, and

$$\begin{aligned} B_i(u) &= u^{p_1} (u-1)^{p_2} \dots (u-i+2)^{p_{i-1}} (u-i+1)^{p_{i+1}} a_i(u) e_i(u-i+1), \\ C_i(u) &= u^{p_1} (u-1)^{p_2} \dots (u-i+1)^{p_i} f_i(u-i+1) a_i(u) \end{aligned}$$

for $i = 1, \dots, n-1$. By the results of [FMO2], $A_i(u)$, $B_i(u)$, and $C_i(u)$ are polynomials in u , and their coefficients are generators of $W(\pi)$. Define the elements $a_r^{(k)}$ for $r = 1, \dots, n$ and $k = 1, \dots, p_1 + \dots + p_r$ by the expansion

$$A_r(u) = u^{p_1 + \dots + p_r} + \sum_{k=1}^{p_1 + \dots + p_r} a_r^{(k)} u^{p_1 + \dots + p_r - k}.$$

Then the elements $a_r^{(k)}$ generate the Gelfand–Tsetlin subalgebra Γ of $W(\pi)$ defined in the Introduction.

Recall some definitions and results from [BK2] regarding representations of $W(\pi)$. Fix an n -tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ of monic polynomials in u , where $\lambda_i(u)$ has degree p_i . We let $L(\lambda(u))$ denote the irreducible highest weight representation of

$W(\pi)$ with the highest weight $\lambda(u)$. Then $L(\lambda(u))$ is generated by a nonzero vector ξ (the highest vector) such that

$$\begin{aligned} B_i(u) \xi &= 0 & \text{for } i = 1, \dots, n-1, & \quad \text{and} \\ u^{p_i} d_i(u) \xi &= \lambda_i(u) \xi & \text{for } i = 1, \dots, n. \end{aligned}$$

Write

$$\lambda_i(u) = (u + \lambda_i^{(1)}) (u + \lambda_i^{(2)}) \dots (u + \lambda_i^{(p_i)}), \quad i = 1, \dots, n.$$

We will be assuming that the parameters $\lambda_i^{(k)}$ satisfy the conditions: for any value $k \in \{1, \dots, p_i\}$ we have

$$\lambda_i^{(k)} - \lambda_{i+1}^{(k)} \in \mathbb{Z}_+, \quad i = 1, \dots, n-1,$$

where \mathbb{Z}_+ denotes the set of nonnegative integers. In this case the representation $L(\lambda(u))$ of $W(\pi)$ is finite-dimensional. We will only consider a certain family of representations of $W(\pi)$ by imposing the condition

$$\lambda_i^{(k)} - \lambda_j^{(m)} \notin \mathbb{Z}, \quad \text{for all } i, j \text{ and all } k \neq m.$$

The *Gelfand–Tsetlin pattern* $\mu(u)$ (associated with the highest weight $\lambda(u)$) is an array of rows $(\lambda_{r1}(u), \dots, \lambda_{rr}(u))$ of monic polynomials in u for $r = 1, \dots, n$, where

$$\lambda_{ri}(u) = (u + \lambda_{ri}^{(1)}) \dots (u + \lambda_{ri}^{(p_i)}), \quad 1 \leq i \leq r \leq n,$$

with $\lambda_{ni}^{(k)} = \lambda_i^{(k)}$, so that the top row coincides with $\lambda(u)$, and

$$\lambda_{r+1,i}^{(k)} - \lambda_{ri}^{(k)} \in \mathbb{Z}_+ \quad \text{and} \quad \lambda_{ri}^{(k)} - \lambda_{r+1,i+1}^{(k)} \in \mathbb{Z}_+$$

for $k = 1, \dots, p_i$ and $1 \leq i \leq r \leq n-1$.

The following theorem was proved in [FMO2]. It will play a key role in the arguments below, as it allows us to realize $W(\pi)$ as a Galois subalgebra; see sec. 3.4. Set $l_{ri}^{(k)} = \lambda_{ri}^{(k)} - i + 1$.

Theorem 2.1. *The representation $L(\lambda(u))$ of the algebra $W(\pi)$ admits a basis $\{\xi_\mu\}$ parameterized by all patterns $\mu(u)$ associated with $\lambda(u)$ such that the action of the generators is given by the formulas*

$$(2.2) \quad A_r(u) \xi_\mu = \lambda_{r1}(u) \dots \lambda_{rr}(u - r + 1) \xi_\mu,$$

for $r = 1, \dots, n$, and

$$(2.3) \quad \begin{aligned} B_r(-l_{ri}^{(k)}) \xi_\mu &= -\lambda_{r+1,1}(-l_{ri}^{(k)}) \dots \lambda_{r+1,r+1}(-l_{ri}^{(k)} - r) \xi_{\mu + \delta_{ri}^{(k)}}, \\ C_r(-l_{ri}^{(k)}) \xi_\mu &= \lambda_{r-1,1}(-l_{ri}^{(k)}) \dots \lambda_{r-1,r-1}(-l_{ri}^{(k)} - r + 2) \xi_{\mu - \delta_{ri}^{(k)}}, \end{aligned}$$

for $r = 1, \dots, n-1$, where $\xi_{\mu \pm \delta_{ri}^{(k)}}$ corresponds to the pattern obtained from $\mu(u)$ by replacing $\lambda_{ri}^{(k)}$ by $\lambda_{ri}^{(k)} \pm 1$, and the vector ξ_μ is considered to be zero, if $\mu(u)$ is not a pattern.

Note that the action of the operators $B_r(u)$ and $C_r(u)$ for an arbitrary value of u can be calculated by the Lagrange interpolation formula.

3. SKEW GROUP STRUCTURE OF FINITE W -ALGEBRAS

3.1. Skew group rings. Let R be a ring, \mathcal{M} a subgroup of $\text{Aut } R$, and $R * \mathcal{M}$ the corresponding skew group ring, i.e., the free left R -module with the basis \mathcal{M} and with the multiplication

$$(r_1 m_1) \cdot (r_2 m_2) = (r_1 r_2^{m_1})(m_1 m_2), \quad m_1, m_2 \in \mathcal{M}, \quad r_1, r_2 \in R.$$

If $x \in R * \mathcal{M}$ and $m \in \mathcal{M}$ then denote by x_m the element of R such that $x = \sum_{m \in \mathcal{M}} x_m m$. Set

$$\text{supp } x = \{m \in \mathcal{M} \mid x_m \neq 0\}.$$

If a finite group G acts by automorphisms on R and by conjugations on \mathcal{M} then G acts on $R * \mathcal{M}$. Denote by $(R * \mathcal{M})^G$ the subring of invariants under this action. Then $x \in (R * \mathcal{M})^G$ if and only if $x_{m^g} = x_m^g$ for $m \in \mathcal{M}, g \in G$.

For $\varphi \in \text{Aut } R$ and $a \in R$ set $H_\varphi = \{h \in G \mid \varphi^h = \varphi\}$ and

$$(3.1) \quad [a\varphi] := \sum_{g \in G/H_\varphi} a^g \varphi^g \in (R * \mathcal{M})^G,$$

where the sum is taken over representatives of the cosets and does not depend on their choice.

3.2. Galois algebras. Let Γ be a commutative domain, K the field of fractions of Γ , $K \subset L$ a finite Galois extension, $G = \text{Gal}(L/K)$ the corresponding Galois group, $\mathcal{M} \subset \text{Aut } L$ a subgroup. Assume that G belongs to the normalizer of \mathcal{M} in $\text{Aut } L$ and $\mathcal{M} \cap G = \{e\}$. Then G acts on the skew group algebra $L * \mathcal{M}$ by automorphisms: $(am)^g = a^g m^g$ where the action on \mathcal{M} is by conjugation. Denote by $(L * \mathcal{M})^G$ the subalgebra of G -invariants in $L * \mathcal{M}$.

Definition 3.1. [FO1] A subring $U \subset (L * \mathcal{M})^G$ finitely generated over Γ is called a *Galois ring over Γ* if $KU = UK = (L * \mathcal{M})^G$.

We will always assume that both Γ and U are \mathbb{k} -algebras and that Γ is noetherian. In this case we will say that a Galois ring U over Γ is a *Galois algebra over Γ* .

Denote by $\bar{\Gamma}$ the integral closure of Γ in L . Let $S_* = \{S_1 \subset S_2 \subset \dots \subset S_N \subset \dots\}$ be an increasing chain of finite sets. Then the growth of S_* is defined as

$$(3.2) \quad \text{growth}(S_*) = \overline{\lim}_{N \rightarrow \infty} \log_N |S_N|.$$

Fix a set of generators $\mathcal{M}_1 = \mathcal{O}_{\varphi_1} \cup \dots \cup \mathcal{O}_{\varphi_n}$ of \mathcal{M} , where $\mathcal{O}_\varphi = \{\varphi^g \mid g \in G\}$. For $N \geq 1$, let \mathcal{M}_N be the set of words $w \in \mathcal{M}$ such that $l(w) \leq N$, where l is the length

of w , that is

$$(3.3) \quad \mathcal{M}_{N+1} = \mathcal{M}_N \bigcup \left(\bigcup_{\varphi \in \mathcal{M}_1} \varphi \cdot \mathcal{M}_N \right).$$

Let $\mathcal{M}_* = \{\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M}_N \subset \dots\}$. Then the growth of \mathcal{M} is by definition $\text{growth}(\mathcal{M}_*)$, we will denote it by $\text{growth}(\mathcal{M})$. For a ring R we will denote by $\text{GKdim } R$ its Gelfand-Kirillov dimension.

Proposition 3.2. [FO1, Theorem 6.1] *Let $U \subset L * \mathcal{M}$ be a Galois algebra over noetherian Γ , \mathcal{M} a group of finite growth such that for every finite dimensional \mathbb{k} -vector space $V \subset \bar{\Gamma}$ the set $\mathcal{M} \cdot V$ is contained in a finite dimensional subspace of $\bar{\Gamma}$. Then*

$$(3.4) \quad \text{GKdim } U \geq \text{GKdim } \Gamma + \text{growth}(\mathcal{M}).$$

3.3. PBW Galois algebras. Let U be an associative algebra over \mathbb{k} , endowed with an increasing exhausting finite-dimensional filtration $\{U_i\}_{i \in \mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \mathbb{k}$. Then $U_i U_j \subset U_{i+j}$ and $\text{gr } U = \bigoplus_{i=0}^{\infty} U_i / U_{i-1}$ is the associated graded algebra. An algebra U is called a *PBW algebra* if $\text{gr } U$ is a commutative affine \mathbb{k} -algebra. In particular, U is a noetherian affine \mathbb{k} -algebra. For PBW algebras we have the following sufficient conditions to be a Galois algebra.

Theorem 3.3. [FO1, Theorem 7.1] *Let U be a PBW algebra generated by the elements u_1, \dots, u_k over Γ , $\text{gr } U$ a polynomial ring in n variables, $\mathcal{M} \subset \text{Aut } L$ a group and $f : U \rightarrow (L * \mathcal{M})^G$ a homomorphism such that $\bigcup_i \text{supp } f(u_i)$ generates \mathcal{M} . If*

$$\text{GKdim } \Gamma + \text{growth}(\mathcal{M}) = n$$

then f is an embedding and U is a Galois algebra over Γ .

3.4. Finite W -algebras as Galois algebras. Now consider the Gelfand–Tsetlin subalgebra Γ of the algebra $W(\pi)$, as defined in sec. 2. Let Λ be the polynomial algebra in the variables x_{ri}^k , $1 \leq i \leq r \leq n$, $k = 1, \dots, p_i$. Consider the \mathbb{k} -homomorphism $\iota : \Gamma \rightarrow \Lambda$ defined by

$$(3.5) \quad \iota(a_r^{(k)}) = \sigma_{r,k}(x_{r1}^1, \dots, x_{r1}^{p_1}, \dots, x_{rr}^1, \dots, x_{rr}^{p_r}), \quad k = 1, \dots, p_1 + \dots + p_r,$$

where $\sigma_{r,j}$ is the j -th elementary symmetric polynomial in $p_1 + \dots + p_r$ variables. The map ι is injective by the theory of symmetric polynomials, and we will identify the elements of Γ with their images in Λ . Let $G = S_{p_1} \times S_{p_1+p_2} \times \dots \times S_{p_1+\dots+p_n}$. Then Γ consists of the invariants in Λ with respect to the natural action of G . Set $\mathcal{L} = \text{Specm } \Lambda$ and identify it with \mathbb{k}^s , $s = np_1 + (n-1)p_2 + \dots + p_n$.

Let $\mathcal{M} \subseteq \mathcal{L}$, $\mathcal{M} \simeq \mathbb{Z}^{(n-1)p_1+\dots+p_{n-1}}$, be the free abelian group generated by the symbols $\delta_{ri}^k \in \mathbb{k}^{(n-1)p_1+\dots+p_{n-1}}$ for $k = 1, \dots, p_i$, $1 \leq i \leq r \leq n-1$. Define an action

of \mathcal{M} on \mathcal{L} by the shifts $\delta_{ri}^k(\ell) := \ell + \delta_{ri}^k$ so that x_{ri}^k is replaced with $x_{ri}^k + 1$, while all other coordinates remain unchanged. The group G acts on \mathcal{L} by permutations and on \mathcal{M} by conjugations.

Let K be the field of fractions of Γ , L the field of fractions of Λ . Then $K \subset L$ is a finite Galois extension with the Galois group G , $K = L^G$. Similarly as above one defines the action of \mathcal{M} on L , the skew group algebra $L * \mathcal{M}$ and its invariant subalgebra $(L * \mathcal{M})^G$.

Recall the polynomials $A_i(u)$, $B_k(u)$, $C_k(u)$ in u , $i = 1, \dots, n$ and $k = 1, \dots, n-1$, with coefficients in $W(\pi)$ which were defined in Section 2.1. Consider the polynomials $\tilde{A}_i(u)$, $\tilde{B}_k(u)$, $\tilde{C}_k(u)$ in u , which are obtained by replacing the nonzero coefficients of the polynomials $A_i(u)$, $B_k(u)$, $C_k(u)$ by independent variables in such a way that the new polynomials have the same degrees as their respective counterparts and the polynomials $\tilde{A}_i(u)$ are monic. Introduce free associative algebra T over \mathbb{k} generated by the coefficients of these polynomials. Let $L[u] * \mathcal{M}$ be the skew group algebra over the ring of polynomials $L[u]$ and e the identity element of \mathcal{M} . Note that $A_i(u) \in L[u] * \mathcal{M}$, $i = 1, \dots, n$. Introduce an algebra homomorphism $t : T \rightarrow L[u] * \mathcal{M}$ by the formulas

$$\begin{aligned} t(\tilde{A}_j(u)) &= A_j(u)e, \\ t(\tilde{B}_r(u)) &= \sum_{(s,j)} X_{rsj}^+[u] \delta_{rj}^s, \\ t(\tilde{C}_r(u)) &= \sum_{(s,j)} X_{rsj}^-[u] (\delta_{rj}^s)^{-1}, \end{aligned}$$

where

$$\begin{aligned} X_{rsj}^+[u] &= -\frac{\prod_{(k,i) \neq (s,j)} (u + x_{ri}^k)}{\prod_{(k,i) \neq (s,j)} (x_{ri}^k - x_{rj}^s)} \prod_{m,q} (x_{r+1,q}^m - x_{rj}^s), \\ X_{rsj}^-[u] &= \frac{\prod_{(k,i) \neq (s,j)} (u + x_{ri}^k)}{\prod_{(k,i) \neq (s,j)} (x_{ri}^k - x_{rj}^s)} \prod_{m,q} (x_{r-1,q}^m - x_{rj}^s), \end{aligned}$$

$j = 1, \dots, r$ and $s = 1, \dots, p_j$. The products (k, i) associated with the variables of the form x_{ri}^k run over the pairs with $i = 1, \dots, r$ and $k = 1, \dots, p_i$.

In the following lemma we use notation (3.1).

Lemma 3.4. *We have*

$$t(\tilde{B}_r(u)) = [X_{r11}^+[u] \delta_{r1}^1], \quad t(\tilde{C}_r(u)) = [X_{r11}^-[u] (\delta_{r1}^1)^{-1}].$$

*In particular, t defines a homomorphism from T to $(L * \mathcal{M})^G$.*

Proof. Note that $H_{\delta_{r1}^1} \subset G$ consists of permutations of G which fix 1, and that X_{r11}^\pm are fixed points of $H_{\delta_{r1}^1}$. Then for $g \in G$, such that $g(1) = p_1 + \dots + p_{i-1} + k$, $0 < k \leq p_i$, the equality $(\delta_{r1}^1)^g = \delta_{ri}^k$ holds and $(X_{r11}^\pm)^g = X_{rki}^\pm$, which implies the statement. \square

Denote by $\pi : T \longrightarrow W(\pi)$ the projection defined by

$$\tilde{A}_r(u) \longmapsto A_r(u), \quad \tilde{B}_r(u) \longmapsto B_r(u), \quad \tilde{C}_r(u) \longmapsto C_r(u).$$

Lemma 3.5. *There exists a homomorphism of algebras $i : W(\pi) \longrightarrow (L * \mathcal{M})^G$, such that the diagram*

$$\begin{array}{ccc} T & \xrightarrow{\pi} & W(\pi) \\ & \searrow \text{t} & \swarrow i \\ & (L * \mathcal{M})^G & \end{array}$$

commutes.

Proof. Let V be a finite-dimensional $W(\pi)$ -module with a basis $\{\xi_\mu\}$. It induces a module structure over T via the homomorphism π . Moreover, due to Theorem 2.1, V has a right module structure over $\text{t}(T) \subset (L * \mathcal{M})^G$. If $z \in T$ and $\text{t}(z) = \sum_{i=1}^s [a_i m_i]$,

$m_i \in \mathcal{M}$, $a_i \in L$, then $\xi_\mu \cdot \text{t}(z) = \sum_{i=1}^s a_i(\mu) \xi_{m_i + \mu}$, where $a_i(\mu)$ means the evaluation

of the rational function $a_i \in L$ in μ . Suppose now that $z \in \text{Ker } \pi$ and consider $\text{t}(z)$. There exists a dense subset $\Omega(z)$ consisting of μ 's, such that ξ_μ is a basis vector of some finite-dimensional $W(\pi)$ -module V and $\xi_\mu \cdot \text{t}(z)$ is defined. Moreover, for any $\mu \in \Omega(z)$, $\xi_\mu \cdot \text{t}(z) = 0$ and hence $a_i(\mu) = 0$ for all i . Since each a_i is a rational function on $\text{Specm } \Lambda$, it implies that $a_i = 0$, and hence $z \in \text{Ker } \text{t}$. Therefore, there exists a homomorphism $i : W(\pi) \longrightarrow (L * \mathcal{M})^G$ such that the diagram commutes. \square

Theorem 3.6. *$W(\pi)$ is a Galois algebra over Γ .*

Proof. First note that $W(\pi)$ is a PBW algebra and $\dim_{\mathbb{k}} \mathcal{M} \cdot v < \infty$ for any $v \in \Lambda$. Also,

$$\begin{aligned} \text{GKdim } W(\pi) &= (2n - 1)p_1 + (2n - 3)p_2 + \dots + 3p_{n-1} + p_n = \\ &= \text{GKdim } \Gamma + \text{growth } \mathcal{M}. \end{aligned}$$

Since $\cup_r \text{supp } \text{t}(\tilde{B}_r(u))$ and $\cup_r \text{supp } \text{t}(\tilde{C}_r(u))$ contain all the generators of the group \mathcal{M} , all conditions of Theorem 3.3 are satisfied. Hence we conclude that $i : W(\pi) \longrightarrow (L * \mathcal{M})^G$ is an embedding and $W(\pi)$ is a Galois algebra over Γ . \square

Recall that a commutative subalgebra A of an associative algebra B is called a *Harish-Chandra subalgebra* if for any $b \in B$, the A -bimodule AbA is finitely generated both as a left and as a right A -module [DFO2].

Corollary 3.7. *Γ is a Harish-Chandra subalgebra of $W(\pi)$.*

Proof. Since $\mathcal{M} \cdot \Lambda \subset \Lambda$ and $W(\pi)$ is a Galois algebra over Γ , the statement follows from [FO1, Proposition 5.2]. \square

Let $\iota : K \rightarrow L$ be a canonical embedding, $\phi \in \text{Aut } L$, $j = \phi\iota$. Consider a (K, L) -bimodule $\tilde{V}_\phi = KvL$, where $av = v\phi(a)$ for all $a \in K$. Let V_ϕ be the set of $\text{St}(j)$ -invariant elements of \tilde{V}_ϕ .

Corollary 3.8. *Let $S = \Gamma \setminus \{0\}$. Then*

(i) *S is an Ore set and*

$$W(\pi)[S^{-1}] \simeq (L * \mathcal{M})^G \simeq [S^{-1}]W(\pi).$$

(ii) *$K \otimes_\Gamma W(\pi) \otimes_\Gamma K \simeq (L * \mathcal{M})^G$ as K -bimodules.*

(iii) *$W(\pi)[S^{-1}] \simeq \bigoplus_{\phi \in \mathcal{M}/G} V_\phi$ as K -bimodules.*

Proof. Follow from Theorem 3.6 and [FO1, Theorem 3.2(5)]. \square

4. NONCOMMUTATIVE NOETHER PROBLEM

If A is a noncommutative domain that satisfies the Ore conditions then it admits the skew field of fractions which we denote $D(A)$.

The n -th Weyl algebra A_n is generated by x_i, ∂_i , $i = 1, \dots, n$ subject to relations

$$(4.1) \quad \begin{aligned} x_i x_j &= x_j x_i, \\ \partial_i \partial_j &= \partial_j \partial_i, \end{aligned}$$

$$(4.2) \quad \partial_i x_j - x_j \partial_i = \delta_{ij}, \quad i, j = 1, \dots, n.$$

This algebra is a simple noetherian domain with the skew field of fractions $D_n = D(A_n)$. The symmetric group S_n acts on A_n and hence on D_n by simultaneous permutations of x_i 's and ∂_i 's.

In this section we prove the noncommutative Noether problem for S_n :

Theorem 4.1. $D_n^{S_n} \simeq D_n$.

4.1. Symmetric differential operators. If $P = \mathbb{k}[x_1, \dots, x_n]$ then we identify the Weyl algebra A_n with the ring of differential operators $\mathcal{D}(P)$ on P by identifying x_i with the operator of multiplication on x_i and ∂_i with the operator of partial derivation by x_i , $i = 1, \dots, n$. If A is a localization of P then $\mathcal{D}(A)$ is generated over A by $\partial_1, \dots, \partial_n$ subject to obvious relations.

It is well known that $A_n^{S_n}$ is not isomorphic A_n and hence $\mathcal{D}(P)^{S_n}$ is not isomorphic to $\mathcal{D}(P^{S_n})$ if $n > 1$. For any $i = 1, \dots, n$ let σ_i denote the i -th symmetric polynomial in the variables x_1, \dots, x_n . Then $P^{S_n} = \mathbb{k}[\sigma_1, \dots, \sigma_n] \subset P$. Set $\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$

and $\Delta = \delta^2 \in P^{S_n}$. Denote by P_Δ and $P_\Delta^{S_n}$ the localizations of corresponding algebras

by the multiplicative set generated by Δ . The canonical embedding $i : P_\Delta^{S_n} \rightarrow P_\Delta$ induces a homomorphism of algebras

$$i_\Delta : \mathcal{D}(P_\Delta)^{S_n} \rightarrow \mathcal{D}(P_\Delta^{S_n}).$$

Let \mathbb{A}^n be the n -dimensional affine space over \mathbb{k} . The algebra $\mathcal{D}(P_\Delta)$ is just the ring of differential operators on $X = \text{Specm } P_\Delta \subset \mathbb{A}^n$ which is open and S_n -invariant. The geometric quotient $X/S_n = \text{Specm } \mathbb{k}[\sigma_1, \dots, \sigma_n]_\Delta$ is rational and the projection $X \rightarrow X/S_n$ is etale. Since the action of S_n on X is free, i_Δ is an isomorphism.

Proposition 4.2. *The following isomorphisms hold*

- (i) *If A is a domain, $S \subset A$ is an Ore subset then $D(A_S) \simeq D(A)$.*
- (ii) $\mathcal{D}(P_\Delta)^{S_n} \simeq (\mathcal{D}(P)^{S_n})_\Delta$.
- (iii) $(P^{S_n})_\Delta \simeq (P_\Delta)^{S_n}$.
- (iv) $\mathcal{D}(P_\Delta)^{S_n} \simeq \mathcal{D}(P^{S_n})_\Delta$.

Proof. The first statement is obvious. Note that $\mathcal{D}(P_S) \simeq \mathcal{D}(P)_S$ for a multiplicative set S , [MCR, Theorem 15.1.25]. If $d \in \mathcal{D}(P_\Delta)^{S_n}$ then $d_1 = \Delta^k d \in \mathcal{D}(P)^{S_n}$ for some $k \geq 0$ implying (ii). The third statement is obvious and (iv) follows from the previous statements. \square

4.2. Proof of Theorem 4.1.

$$\begin{aligned} D_n^{S_n} &\simeq D^{S_n}(\mathcal{D}(P)) \simeq D^{S_n}(\mathcal{D}(P)_\Delta) \simeq D^{S_n}(\mathcal{D}(P_\Delta)) \simeq D(\mathcal{D}^{S_n}(P_\Delta)) \simeq D(\mathcal{D}((P_\Delta)^{S_n})) \\ &\simeq D(\mathcal{D}(\mathbb{k}[\sigma_1, \dots, \sigma_n]_\Delta)) \simeq D((\mathcal{D}(\mathbb{k}[\sigma_1, \dots, \sigma_n])_\Delta) \simeq D(\mathcal{D}(\mathbb{k}[\sigma_1, \dots, \sigma_n])) \simeq D_n. \end{aligned}$$

Hence $D_n^{S_n} \simeq D_n$.

5. GELFAND-KIRILLOV CONJECTURE

Since $W(\pi)$ is a noetherian integral domain with a polynomial graded algebra, then it satisfies the Ore conditions by the Goldie theorem. Let $D_\pi(n) = D(W(\pi))$ be the skew field of fractions of $W(\pi)$. Recall the structure of $W(\pi)$ as a Galois algebra over Γ : $W(\pi) \subset (L * \mathcal{M})^G$, where L is a field of rational functions in x_{ij}^k , $j = 1, \dots, i, k = 1, \dots, p_i, i = 1, \dots, n$. Then $D_\pi(n) \simeq D((L * \mathcal{M})^G)$. Moreover, we will see below that $L * \mathcal{M}$ has a skew field of fractions and thus $D_\pi(n) \simeq D(L * \mathcal{M})^G$ [Fa, Theorem 1]. Since Γ is a Harish-Chandra subalgebra (Corollary 3.7) then by [FO1, Theorem 4.1], we have

Proposition 5.1. *The center \mathcal{Z} of $D_\pi(n)$ is isomorphic to $K^\mathcal{M}$.*

Let Λ be the polynomial ring in variables x_{ij}^k , $j = 1, \dots, i, k = 1, \dots, p_j, i = 1, \dots, n$. Denote by L_i (respectively Λ_i) the field of rational functions (respectively the polynomial ring) in x_{ij}^k with fixed i . Then

$$\Lambda * \mathcal{M}^G \simeq \bigotimes_{i=1}^{n-1} (\Lambda_i * \mathbb{Z}^{p_1 + \dots + p_i})^{S_{p_1 + \dots + p_i}} \otimes \Lambda_n^{S_{p_1 + \dots + p_n}}.$$

Proposition 5.2. *For every $i = 1, \dots, n$*

$$D(L_i * \mathbb{Z}^{p_1+\dots+p_i}) \simeq D(A_{p_1+\dots+p_i}(\mathbb{k})).$$

Proof. Let $B_i = \mathbb{k}[t_1, \dots, t_i] * \mathbb{Z}^i$, where \mathbb{Z}^i is generated by σ_k , $k = 1, \dots, i$ and $\sigma_k(t_m) = t_m - \delta_{km}$. Then B_i is isomorphic to the localization \mathcal{A}_i of the i -th Weyl algebra with respect to x_1, \dots, x_i . This isomorphism is given as follows:

$$x_k \mapsto \sigma_k, \quad \partial_k \mapsto t_k \sigma_k^{-1}.$$

Hence, a subring $\Lambda_i * \mathbb{Z}^{p_1+\dots+p_i}$ of $L_i * \mathbb{Z}^{p_1+\dots+p_i}$ is isomorphic to a localization of $A_{p_1+\dots+p_i}(\mathbb{k})$ which implies the statement. \square

Since $D(A_k)^{S_k} \simeq D(A_k^{S_k})$ then we have the isomorphism

$$D((L * \mathcal{M})^G) = D((\Lambda * \mathcal{M})^G) \simeq \otimes_{i=1}^{n-1} D((A_{p_1+\dots+p_i}(\mathbb{k}))^{S_{p_1+\dots+p_i}} \otimes D(T_n)),$$

where $T_n = \Lambda_n^{S_{p_1+\dots+p_n}}$ is a polynomial ring isomorphic Λ_n . Moreover, by applying Theorem 4.1 we have the isomorphism

$$D((L * \mathcal{M})^G) \simeq D(A_{(n-1)p_1+\dots+p_{n-1}}(\mathbb{k}) \otimes D(T_n)).$$

Since $D(T_n)$ is a pure transcendental extension of \mathbb{k} of degree $p_1 + \dots + p_n$, and since $D((L * \mathcal{M})^G) \simeq D(W(\pi))$, we have thus proved the Gelfand-Kirillov conjecture (Theorem I):

$$D(W(\pi)) \simeq D(A_{(n-1)p_1+\dots+p_{n-1}}(D(T_n))) = D_{k,m},$$

$k = (n-1)p_1 + \dots + p_{n-1}$, $m = p_1 + \dots + p_n$.

Recall that the *Miura transform* [BK2] is an injective homomorphism

$$\tau : W(\pi) \rightarrow \otimes_{i=1}^l U(\mathfrak{gl}_{q_i}).$$

Observe that $D(\otimes_{i=1}^l U(\mathfrak{gl}_{q_i})) \simeq D_{k,m}$, since $k = \sum_{i=1}^l q_i(q_i - 1)/2$ and $m = \sum_{i=1}^l q_i$. Hence we have proved the following corollary.

Corollary 5.3. *The Miura transform extends to an isomorphism of the corresponding skew fields of fractions.*

6. FIBERS OF CHARACTERS

6.1. Galois orders. Let $U \subset (L * \mathcal{M})^G$ be a Galois ring over an integral domain Γ .

Definition 6.1. [FO1] A Galois ring U over Γ is called a *Galois order* if for any finite dimensional right (respectively left) K -subspace $V \subset U[S^{-1}]$ (respectively $V \subset [S^{-1}]U$), $V \cap U$ is a finitely generated right (respectively left) Γ -module.

A concept of a Galois order over Γ is a natural noncommutative generalization of a classical notion of Γ -order in skew group ring $(L * \mathcal{M})^G$. If Γ is a noetherian \mathbb{k} -algebra then a Galois order over Γ will be called an *integral Galois algebra*. Note that in

particular a Galois ring U over Γ is a Galois order if U is a projective right and left Γ -module.

The following criterion for Galois orders was established in [FO1, Corollary 5.6].

Proposition 6.2. *Let $U \subset L * \mathcal{M}$ be a Galois algebra over a noetherian normal \mathbb{k} -algebra Γ . Then the following statements are equivalent*

- (i) U is an integral Galois algebra over Γ .
- (ii) Γ is a Harish-Chandra subalgebra and, if for $u \in U$ there exists a nonzero $\gamma \in \Gamma$ such that $\gamma u \in \Gamma$ or $u\gamma \in \Gamma$, then $u \in \Gamma$.

Suppose now that U is a PBW Galois algebra over Γ with the polynomial associated graded algebra $\text{gr } U = A$. Then both U and A are endowed with degree function \deg with obvious properties. For $u \in U$ denote by $\bar{u} \in A$ the corresponding homogeneous element. Also denote by $\text{gr } \Gamma$ the image of Γ in A . Then we have the following graded version of Proposition 6.2.

Lemma 6.3. *Let $U \subset L * \mathcal{M}$ be a PBW Galois algebra over a noetherian normal \mathbb{k} -algebra Γ with a polynomial graded algebra $\text{gr } U$. Then the following statements are equivalent*

- (i) U is an integral Galois algebra over Γ .
- (ii) Γ is a Harish-Chandra subalgebra and for $\gamma, \gamma' \in \Gamma \setminus \{0\}$ it follows from $\bar{\gamma}' = \bar{\gamma}a, a \in A$ that $a \in \text{gr } \Gamma$.

Proof. Suppose $\gamma' = \gamma u \neq 0$, $\gamma', \gamma \in \Gamma$, $u \in U \setminus \Gamma$ and $\deg \gamma'$ is the minimal possible. Then $\bar{\gamma}' = \bar{\gamma}\bar{u} \neq 0$ in A . By the assumption $\bar{u} = \bar{\gamma}''$ for some $\gamma'' \in \Gamma$ and hence either $\gamma'' = u$, or $\gamma_2 = \gamma u_1 \in \Gamma$, where $u_1 = u - \gamma''$, $\gamma_2 = \gamma' - \gamma\gamma''$. Since in the second case $\deg \gamma_2 < \deg \gamma_1$ this contradicts the minimality assumption. Therefore, $\gamma'' = u \in \Gamma$. The case $\gamma' = u\gamma \neq 0$ is considered analogously. Hence the statement (6.2) of Proposition 6.2 holds, which implies the integrality of the Galois algebra U . \square

Representation theory of Galois algebras was developed in [FO2]. For $\mathbf{m} \in \text{Specm } \Gamma$ denote by $F(\mathbf{m})$ the fiber of \mathbf{m} consisting of isomorphism classes of irreducible Gelfand-Tsetlin (with respect to Γ) U -modules M with $M(\mathbf{m}) \neq 0$.

Let E be the integral extension of Γ such that $\Gamma = E^G$ and assume that Γ is noetherian. Then the fibers of the surjective map $\varphi : \text{Specm } E \rightarrow \text{Specm } \Gamma$ are finite. Let $\mathbf{m} \in \text{Specm } \Gamma$ and $l_{\mathbf{m}} \in \text{Specm } E$ such that $\varphi(l_{\mathbf{m}}) = \mathbf{m}$. Denote

$$\text{St}_{\mathcal{M}}(\mathbf{m}) = \{x \in \mathcal{M} | x \cdot l_{\mathbf{m}} = l_{\mathbf{m}}\}.$$

Clearly the set $\text{St}_{\mathcal{M}}(\mathbf{m})$ does not depend on the choice of $l_{\mathbf{m}}$.

Theorem 6.4. [FO2, Theorem A] *Let U be an integral Galois algebra over noetherian Γ , $\mathbf{m} \in \text{Specm } \Gamma$. If the set $\text{St}_{\mathcal{M}}(\mathbf{m})$ is finite then the fiber $F(\mathbf{m})$ is non-trivial and finite.*

6.2. Finite W -algebras as integral Galois algebras. In this section we show that $W(\pi)$ is an integral Galois algebra over Γ .

Following [BK2, Section 2.2], for $1 \leq i \leq j \leq n$ define the higher root elements $e_{ij}^{(r)}$ and $f_{ji}^{(r)}$ of $W(\pi)$ inductively by the formulas $e_{i,i+1}^{(r)} = e_i^{(r)}$ for $r \geq p_{i+1} - p_i + 1$,

$$e_{ij}^{(r)} = [e_{i,j-1}^{(r-p_j+p_{j-1})}, e_{j-1}^{(p_j-p_{j-1}+1)}] \quad \text{for } r \geq p_j - p_i + 1,$$

and

$$f_{i+1,i}^{(r)} = f_i^{(r)}, \quad f_{j,i}^{(r)} = [f_{j-1}^{(1)}, f_{j-1,i}^{(r)}] \quad \text{for } r \geq 1.$$

Furthermore, set

$$e_{ij}(u) = \sum_{r=p_j-p_i+1}^{\infty} e_{ij}^{(r)} u^{-r}, \quad f_{ji}(u) = \sum_{r=1}^{\infty} f_{ji}^{(r)} u^{-r},$$

and define a power series

$$t_{ij}(u) = \sum_{r \geq 0} t_{ij}^{(r)} u^{-r} = \sum_{k=1}^{\min\{i,j\}} f_{ik}(u) d_k(u) e_{kj}(u)$$

for some elements $t_{ij}^{(r)} \in W(\pi)$. Due to [BK2, Lemma 3.6], an ascending filtration on $W(\pi)$ can be defined by setting $\deg t_{ij}^{(k)} = k$. Let $\overline{W}(\pi) = \text{gr } W(\pi)$ denote the associated graded algebra and let $\bar{t}_{ij}^{(r)}$ denote the image of $t_{ij}^{(r)}$ in the r th component of $\text{gr } W(\pi)$. Then $\overline{W}(\pi)$ is a polynomial algebra in the variables

$$\bar{t}_{ij}^{(r)} \quad \text{with } i \geq j, \quad 1 \leq r \leq p_j \quad \text{and} \quad \bar{t}_{ij}^{(r)} \quad \text{with } i < j, \quad p_j - p_i + 1 \leq r \leq p_j.$$

By [BK2, Theorem 3.5], the series

$$T_{ij}(u) = u^{p_j} t_{ij}(u), \quad 1 \leq i, j \leq n,$$

are polynomials in u . Introduce the matrix $T(u) = (T_{ij}(u - j + 1))_{i,j=1}^n$ and consider its *column determinant*

$$(6.1) \quad \text{cdet } T(u) = \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot T_{\sigma(1)1}(u) T_{\sigma(2)2}(u-1) \dots T_{\sigma(n)n}(u-n+1).$$

This is a polynomial in u , and the coefficients $d_s \in W(\pi)$ of the powers $u^{p_1+\dots+p_n-s}$, $s = 1, \dots, p_1 + \dots + p_n$ are algebraically independent generators of the center of $W(\pi)$; see [BB].

For $F = \sum_i f_i u^i \in W(\pi)[u]$ denote $\overline{F} = \sum_i \bar{f}_i u^i \in \overline{W}(\pi)[u]$. Also we denote $X_{ij}^k = \bar{t}_{ij}^{(k)}$ for $k \geq 1$ and $X_{ij}^0 = \delta_{ij}$. Set $X_{ij}(u) = \overline{T}_{ij}(u)$ and $X(u) = (X_{ij}(u))_{i,j=1}^n$. Since $\overline{T_{ij}(u - \lambda)} = X_{ij}(u)$ for any $\lambda \in \mathbb{k}$, one can easily check that $\text{gr cdet } T(u) = \det X(u)$.

Then

$$(6.2) \quad \bar{d}_s = \sum_{k_1+\dots+k_n=s} \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot X_{\sigma(1)1}^{k_1} \dots X_{\sigma(n)n}^{k_n}$$

is just the coefficient of $u^{p_1+\dots+p_n-s}$ in $\det X(u)$.

Fix r , $1 \leq r \leq n$ and consider $X_r(u) = (X_{ij}(u))_{i,j=1}^r$. Then

$$(6.3) \quad d_{rs} = \sum_{k_1+\dots+k_r=s} \sum_{\sigma \in S_r} \operatorname{sgn} \sigma \cdot X_{\sigma(1)1}^{k_1} \dots X_{\sigma(r)r}^{k_r}$$

is the coefficient of $u^{p_1+\dots+p_r-s}$ in $\det X_r(u)$ and the elements

$$\{d_{rs}, \quad r = 1, \dots, n, \quad s = 1, \dots, p_1 + \dots + p_r\}$$

are the generators of the algebra $\operatorname{gr} \Gamma$.

We will use the idea of a weighted polynomial order on $\overline{W}(\pi)$ ([GP]). Let $S = \{X_{ij}^k \mid i, j = 1, \dots, n; k = 1, \dots, p_j\}$, $w : S \rightarrow \mathbb{N}$ be a function. Define the degree of each variable X_{ij}^k as $w(X_{ij}^k)$ and then define the degree of any monomial in these variables as the sum of the degrees of the variables occurring in the monomial. We will denote this degree associated with w by \deg_w . It coincides with the usual polynomial degree if $w(X_{ij}^k) = 1$ for all i, j, k . Also it coincides with the degree in $\overline{W}(\pi)$ if $w(X_{ij}^k) = k$ for all i, j . Fixing an order on S we define a lexicographic order on the monomials. For the monomials m_1 and m_2 define $m_1 >_w m_2$ provided $\deg_w(m_1) > \deg_w(m_2)$ or $\deg_w(m_1) = \deg_w(m_2)$ and $m_1 > m_2$ in the lexicographical order. It allows us to define the leading monomial of $f \in \overline{W}(\pi)$ with respect to w . If m is a leading monomial of f then set $\operatorname{lm}(f) = m$. The coefficient of m in f we denote by $\operatorname{lc}(m) = \operatorname{lc}(f)$. Note that the weighted polynomial order \deg_w and the concepts of $\operatorname{lm}(f)$ and $\operatorname{lc}(f)$ naturally extend to $W(\pi)$ and Γ .

Lemma 6.5. (1) *There exists a weight function w such that for any $r = 1, \dots, n$ and $s = 1, \dots, p_1 + \dots + p_r$ the leading monomial m_{rs} of d_{rs} contains a variable $X(r, s) = X_{ij}^k$, ($i, j, k = i(r, s), j(r, s), k(r, s)$) which does not enter in leading monomials $m_{r's'}$ for $(r', s') \neq (r, s)$. Besides, the variable $X(r, s)$ enters in m_{rs} in degree 1.*

(2) *For any $\gamma \in \operatorname{gr} \Gamma$ there exist $f = \prod_{r,s} d_{rs}^{k_{r,s}}$ and $\lambda \in \mathbb{K}$ such that $\gamma >_w (\gamma - \lambda f)$.*

(3) *If $\gamma, \gamma_1 \in \operatorname{gr} \Gamma$ and $\operatorname{lm}(\gamma_1) \mid \operatorname{lm}(\gamma)$, then there exists $\gamma_2 \in \operatorname{gr} \Gamma$, such that*

$$\operatorname{lm}(\gamma) = \operatorname{lm}(\gamma_1) \operatorname{lm}(\gamma_2).$$

Proof. Define a function v on S with values in \mathbb{Z} satisfying the following conditions:

- (i) $v(X_{i+1,i}^{p_i}) = i + 1$, $i = 1, \dots, n - 1$;
- (ii) $v(X_{ij}^k) = -N$, where $N > 2n^2$, if $i < j$, $i, j = 1, \dots, n$;
- (iii) $v(X_{ii}^k)$ are significantly smaller than those above,
 $v(X_{ii}^k) > v(X_{jj}^l)$ if $i > j$ or $i = j, k > l$;
- (iv) For $i - j \geq 2$ or $j = i - 1, k < p_{i-1}$ the values $v(X_{ij}^k)$ are negative and its absolute values are significantly larger than the absolute values of those above.

In particular, if a monomial m from (6.3) contains X_{ij}^k satisfying (iv), then $v(m) < v(m')$ for any m' which does not contain such variable.

First we will construct a required monomial for the weight function v . Fix $r \in \{1, \dots, n\}$ and $s \in \{1, \dots, p_1 + \dots + p_r\}$.

If $s \leq p_r$ then set

$$y_{r,s} = X_{rr}^s.$$

Suppose $p_r < s \leq p_r + p_{r-1}$ and consider

$$y_{r,s} = X_{r,r-1}^{p_{r-1}} X_{r-1,r}^{s-p_{r-1}}.$$

Note that $s - p_{r-1} \leq p_r$ and $p_r - p_{r-1} < s - p_{r-1}$. Generalizing, suppose

$$p_r + \dots + p_{r-t+1} < s \leq p_r + \dots + p_{r-t},$$

for some t , $2 \leq t \leq r-1$ (such t is uniquely defined for given r, s). In this case set

$$y_{r,s} = X_{r,r-1}^{p_{r-1}} X_{r-1,r-2}^{p_{r-2}} \dots X_{r-t+1,r-t}^{p_{r-t}} X_{r-t,r}^k,$$

where $k = s - (p_{r-1} + \dots + p_{r-t})$. We have $p_r - p_{r-t} < k \leq p_r$.

It is easy to see that the defined monomials $y_{r,s}$ belong to d_{rs} . Moreover, any other monomial in d_{rs} has weight strictly smaller than $y_{r,s}$. Indeed, the condition (iv) shows that if a leading monomial in d_{rs} contains X_{ij}^k , where $i > j$, then $i = j+1$ and $k = p_j$. Hence $y_{r,s}$ is the leading monomial of d_{rs} if $s \leq p_r$. For the case $s > p_r$ the conditions (iii) and (iv) show that the leading monomial of d_{rs} contains only $X_{i+1,i}^{p_i}$ and X_{ij}^b for $i < j$. By the condition (i) we have

$$v(X_{r,r-1}^{p_{r-1}}) > v(X_{r-1,r-2}^{p_{r-2}}) > \dots > v(X_{21}^{p_1})$$

and hence $X_{i+1,i}^{p_i}$ will enter the leading monomial with a largest possible value of i . It is clear now that $y_{r,s}$ is the leading monomial of d_{rs} .

Now choose a sufficiently large integer $l > 0$ such that $v(x_{ij}^k) + kl \in \mathbb{N}$ for all possible i, j, k . We can define the required function $w : S \rightarrow \mathbb{N}$ by $w(x_{ij}^k) = v(x_{ij}^k) + kl$. Since d_{rs} are homogeneous, their leading monomials do not change after the shift of gradation. We conclude that with respect to the function w , the elements

$$\{y_{r,s} \mid r = 1, \dots, n; s = 1, \dots, p_1 + \dots + p_r\}$$

are the leading monomials of the generators of $\text{gr}\Gamma \subset \overline{W}(\pi)$.

Note that $y_{r,s} \neq y_{r',s'}$ for different pairs r, s and r', s' . Given r and s let t be such that $0 \leq t \leq r-1$ and $p_r + \dots + p_{r-t+1} < s \leq p_r + \dots + p_{r-t}$. Set $X(r, s) = X_{r-t,r}^k$, $k = s - (p_{r-1} + \dots + p_{r-t})$ if $t > 0$, and $X(r, s) = X_{rr}^s$ if $t = 0$. Then $X(r, s)$ satisfies (1).

For any $\gamma \in \text{gr}\Gamma$, the number of occurrences of d_{rs} in $\text{lm}(\gamma)$ equals the number of occurrences of $X(r, s)$ in $\text{lm}(\gamma)$. Denote this number by k_{rs} and set $f = \prod_{r,s} d_{rs}^{k_{rs}}$. Let $\lambda = \text{lc}(\gamma)$. Then

$$\deg_w(\gamma) > \deg_w(\gamma - \lambda f),$$

implying (2) and thus (3). □

Theorem 6.6. *Let $\Gamma \subset W(\pi)$ be the Gelfand-Tsetlin subalgebra of $W(\pi)$. Then $W(\pi)$ is an integral Galois algebra over Γ .*

Proof. First recall that Γ is a Harish-Chandra subalgebra. Assume that $\gamma a \in \text{gr } \Gamma$ for some $\gamma \in \text{gr } \Gamma$ and $a \in \bar{W}(\pi)$. Let w be the function constructed in Lemma 6.5. Then $\text{lm}(\gamma a) = \text{lm}(\gamma) \text{lt}(a)$. Following Lemma 6.5, (3), there exists $\gamma' \in \text{gr } \Gamma$, such that $\text{lm}(\gamma') = \text{lt}(a)$. Consider $a' = a - \gamma'$. Then we have $\gamma a' \in \text{gr } \Gamma$ and $\deg_w(a') < \deg_w(a)$. Applying induction in $\deg_w(a)$ we conclude that $a' \in \text{gr } \Gamma$ and hence $a \in \text{gr } \Gamma$. It remains to apply Lemma 6.3. \square

Since $W(\pi)$ is integral Galois algebra over Γ and Γ is noetherian then $W(\pi) \cap K \subset L$ is an integral extension of Γ by [FO1, Theorem 5.2]. Since $W(\pi)$ is a Galois algebra over Γ then $K \cap W(\pi)$ is a maximal commutative \mathbb{k} -subalgebra in $W(\pi)$ by [FO1, Theorem 4.1]. But Γ is integrally closed in K . Hence we obtain

Corollary 6.7. *Γ is a maximal commutative subalgebra in $W(\pi)$.*

6.3. Proof of Theorem II. We are now in a position to prove our main result on Gelfand-Tsetlin modules announced in Introduction. Since the Gelfand-Tsetlin subalgebra is a polynomial ring, $W(\pi)$ is integral Galois algebra by Theorem 6.6, and since for any $\mathbf{m} \in \text{Specm } \Gamma$ the set $\text{St}_{\mathcal{M}}(\mathbf{m})$ is finite, then Theorem II follows immediately from Theorem 6.4. Therefore every character $\chi : \Gamma \rightarrow \mathbb{k}$ of the Gelfand-Tsetlin subalgebra defines an irreducible Gelfand-Tsetlin module which is a quotient of $W(\pi)/W(\pi)\mathbf{m}$, $\mathbf{m} = \text{Ker } \chi$. Of course different characters can give isomorphic irreducible modules. In such case we say that these characters are equivalent. Therefore we obtain a classification of irreducible Gelfand-Tsetlin modules by the equivalence classes of characters of Γ up to a certain finiteness. This finiteness corresponds to finite fibers of irreducible Gelfand-Tsetlin modules with a given character of Γ .

7. CATEGORY OF GELFAND-TSETLIN MODULES

For a Γ -bimodule V denote by ${}_n\hat{V}_{\mathbf{m}}$ the I -adic completion of $\Gamma \otimes_{\mathbb{k}} \Gamma$ -module V , where $I \subset \Gamma \otimes \Gamma$ is a maximal ideal $I = \mathbf{n} \otimes \Gamma + \Gamma \otimes \mathbf{m}$, that is

$${}_n\hat{V}_{\mathbf{m}} = \varprojlim_{n,m} {}_nV_{\mathbf{m}^m},$$

here ${}_nV_{\mathbf{m}^m} = V/(\mathbf{n}^n V + V\mathbf{m}^m)$. Let $F(W(\pi))$ be the set of finitely generated Γ -subbimodules in $W(\pi)$.

Define a category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ with the set of objects $\text{Ob } \mathcal{A} = \text{Specm } \Gamma$ and with the space of morphisms $\mathcal{A}(\mathbf{m}, \mathbf{n})$ from \mathbf{m} to \mathbf{n} , where

$$\mathcal{A}(\mathbf{m}, \mathbf{n}) = \varinjlim_{V \in F(W(\pi))} {}_n\hat{V}_{\mathbf{m}}.$$

Consider the completion $\Gamma_{\mathbf{m}} = \varprojlim_n \Gamma/\mathbf{m}^n$ of Γ by the ideal $\mathbf{m} \in \text{Specm } \Gamma$. Then the space $\mathcal{A}(\mathbf{m}, \mathbf{n})$ has a natural structure of $(\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{m}})$ -bimodule. The category \mathcal{A} is naturally endowed with the topology of the inverse limit. Consider the category $\mathcal{A}\text{-mod}_d$ of continuous functors $M : \mathcal{A} \rightarrow \mathbb{k}\text{-mod}$, [DFO2, Section 1.5], where $\mathbb{k}\text{-mod}$ is endowed with the discrete topology.

Let $\mathbb{H}(W(\pi), \Gamma)$ denote the category of Gelfand-Tsetlin modules with respect to the Gelfand-Tsetlin subalgebra Γ for finite W -algebra $W(\pi)$. Since Γ is a Harish-Chandra subalgebra by Corollary 3.7 then by [DFO2, Theorem 17] (see also [FO2, Theorem 3.2]), the categories $\mathcal{A}\text{-mod}_d$ and $\mathbb{H}(W(\pi), \Gamma)$ are equivalent.

A functor that determines this equivalence can be defined as follows. For $N \in \mathcal{A}\text{-mod}_d$ set

$$\mathbb{F}(N) = \bigoplus_{\mathbf{m} \in \text{Specm } \Gamma} N(\mathbf{m})$$

and for $x \in N(\mathbf{m})$, $a \in U$ set

$$ax = \sum_{\mathbf{n} \in \text{Specm } \Gamma} a_{\mathbf{n}} x,$$

where $a_{\mathbf{n}}$ is the image of a in $\mathcal{A}(\mathbf{m}, \mathbf{n})$. If $f : M \rightarrow N$ is a morphism in $\mathcal{A}\text{-mod}_d$ then set $\mathbb{F}(f) = \bigoplus_{\mathbf{m} \in \text{Specm } \Gamma} f(\mathbf{m})$. Hence we obtain a functor

$$\mathbb{F} : \mathcal{A}\text{-mod}_d \rightarrow \mathbb{H}(W(\pi), \Gamma).$$

For $\mathbf{m} \in \text{Specm } \Gamma$ denote by $\hat{\mathbf{m}}$ the completion of \mathbf{m} . Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \text{Specm } \Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$.

Let $\mathbb{H}W(W(\pi), \Gamma)$ be the full subcategory of *weight* Gelfand-Tsetlin modules M such that $\mathbf{m}v = 0$ for any $v \in M(\mathbf{m})$. Clearly, the categories $\mathbb{H}W(W(\pi), \Gamma)$ and $\mathcal{A}_W\text{-mod}$ are equivalent.

For a given $\mathbf{m} \in \text{Specm } \Gamma$ denote by $\mathcal{A}_{\mathbf{m}}$ the indecomposable block of the category \mathcal{A} which contains \mathbf{m} .

An embedding $\iota : \Gamma \rightarrow \Lambda$ induces an epimorphism

$$\iota^* : \mathcal{L} \rightarrow \text{Specm } \Gamma.$$

Denote by $\tilde{\Omega} \subset \mathcal{L}$ the set of generic parameters $\mu = (\mu_{ij}^k, i = 1, \dots, n; j = 1, \dots, i; k = 1, \dots, p)$ such that

$$\mu_{ij}^k - \mu_{i,s}^q \notin \mathbb{Z}, \quad \mu_{r+1,j}^{(m)} - \mu_{ri}^{(k)} \notin \mathbb{Z}$$

for all r, i, j, m, k .

Theorem 7.1. *Let $\mathbf{m} \in \text{Specm } \Gamma$, $\mu \in (\iota^*)^{-1}(\mathbf{m})$. Suppose $\mu \in \tilde{\Omega}$. Then*

- (i) *All objects of $\mathcal{A}_{\mathbf{m}}$ are isomorphic and for every $\mathbf{n} \in \mathcal{A}_{\mathbf{m}}$,*

$$\mathcal{A}(\mathbf{n}, \mathbf{n}) \simeq \hat{\Gamma}_{\mathbf{n}}.$$

(ii) Let $M_{\mathbf{m}} = \mathcal{A}_{\mathbf{m}}/\mathcal{A}_{\mathbf{m}}\hat{\mathbf{m}}$. Then there is a canonical isomorphism

$$\mathbb{F}(M_{\mathbf{m}}) \simeq W(\pi)/W(\pi)\mathbf{m}.$$

(iii) The category $\mathbb{H}(W(\pi), \Gamma, \mathbf{m})$ which consists of modules whose support belongs to $\mathcal{A}_{\mathbf{m}}$, is equivalent to the extension category generated by module $\mathbb{F}(M_{\mathbf{m}})$. Moreover, this category is equivalent to the category $\hat{\Gamma}_{\mathbf{m}}\text{-mod}$.

Proof. Since \mathcal{M} acts freely on $\tilde{\Omega}$ and $\mathcal{M} \cdot \mu \cap G \cdot \mu = \{\mu\}$ all statements follow from Theorem 6.6 and [FO2, Theorem 5.3, Theorem B]. \square

Since for \mathbf{m} from Theorem 7.1, $\hat{\Gamma}_{\mathbf{m}}$ is isomorphic to the algebra of formal power series in $\text{GKdim } \Gamma$ variables, we immediately obtain the statements of Theorem III.

8. W -ALGEBRAS ASSOCIATED WITH \mathfrak{gl}_2

In this section we consider the case of W -algebras associated with \mathfrak{gl}_2 : $W(\pi)$, where π has rows (p_1, p_2) . We will show that $W(\pi)$ is free over the Gelfand-Tsetlin subalgebra. A particular case $p_1 = p_2$ was considered in [FMO1].

The shifted Yangian $W(\pi)$ is generated by $t_{11}^{(k)}, t_{21}^{(k)}, k = 1, \dots, p_1, t_{22}^{(r)}, r = 1, \dots, p_2$ and $t_{12}^{(m)}, m = p_2 - p_1 + 1, \dots, p_2$.

We will denote by $\bar{t}_{11}^{(k)}, \bar{t}_{21}^{(k)}, \bar{t}_{22}^{(k)}, \bar{t}_{12}^{(k)}$ the images of the generators of $W(\pi)$ in the graded algebra $\bar{W}(\pi)$.

Let

$$\begin{aligned} T_{11}(u) &= \sum_{i=0}^{p_1} t_{11}^{(i)} u^{p_1-i}, & T_{22}(u) &= \sum_{i=0}^{p_2} t_{22}^{(i)} u^{p_2-i}, \\ T_{21}(u) &= \sum_{i=1}^{p_1} t_{21}^{(i)} u^{p_1-i}, & T_{12}(u) &= \sum_{i=1}^{p_1} t_{12}^{(s+i)} u^{p_1-i} \end{aligned}$$

and

$$D_1(u) = T_{11}(u), \quad D_2(u) = T_{11}(u+1)T_{22}(u) - T_{21}(u+1)T_{12}(u).$$

The coefficients $d_1^{(1)}, \dots, d_{p_1}^{(1)}$ of $D_1(u)$ and $d_1^{(2)}, \dots, d_{p_1+p_2}^{(2)}$ of $D_2(u)$ are generators of the Gelfand-Tsetlin subalgebra Γ . Denote by $\bar{d}_i^{(j)}$ their images in the graded algebra.

Recall that a sequence x_1, \dots, x_n of elements of some commutative ring R is called *regular* if, for all $i = 1, \dots, n$, the multiplication by x_i is injective on

$$R / \langle x_1, \dots, x_{i-1} \rangle R$$

and $R / \langle x_1, \dots, x_n \rangle R \neq 0$.

Since $W(\pi)$ is a *special filtered* algebra in the sense of [FO3], by [FO3, Theorem 1.1] we only need to show that $\bar{d}_1^{(1)}, \dots, \bar{d}_{p_1}^{(1)}, \bar{d}_1^{(2)}, \dots, \bar{d}_{p_1+p_2}^{(2)}$ is a regular sequence in $\bar{W}(\pi)$.

We will use the following standard result.

Lemma 8.1. *A sequence of the form $x_1, \dots, x_r, y_1, \dots, y_t$, where y_1, \dots, y_t are homogeneous elements of $A = \mathbb{k}[x_1, \dots, x_q]$, $q > r$, is regular in A if and only if the sequence $\tilde{y}_1, \dots, \tilde{y}_t$ is regular in $\mathbb{k}[x_{r+1}, \dots, x_q]$, where $\tilde{y}_i(x_{r+1}, \dots, x_q) = y_i(0, \dots, 0, x_{r+1}, \dots, x_q)$.*

Applying Lemma 8.1 we reduce the problem to the problem of regularity of the sequence of images of $\bar{d}_1^{(2)}, \dots, \bar{d}_{p_1+p_2}^{(2)}$ in $\bar{W}(\pi)/(D_1(u))$. Consider the first p_2 elements in this sequence. Then the image of $\bar{d}_i^{(2)}$ coincides with $\bar{t}_{22}^{(i)}$, $i = 1, \dots, p_2$. Hence, applying again Lemma 8.1 we reduce the problem to the regularity of the sequence of images of $\bar{d}_{p_2+1}^{(2)}, \dots, \bar{d}_{p_1+p_2}^{(2)}$ in $\bar{W}(\pi)/(T_{11}(u), T_{22}(u))$. Denote these elements by $z_{p_2+1}, \dots, z_{p_1+p_2}$.

Consider the restricted Yangian $Y_{p_2}(\mathfrak{gl}_2)$ of level p_2 (see [FMO1]) generated by the coefficients of the polynomials

$$\begin{aligned} T'_{11}(u) &= \sum_{i=0}^{p_2} t_{11}^{(i)} u^{p_2-i}, & T'_{22}(u) &= \sum_{i=0}^{p_2} t_{22}^{(i)} u^{p_2-i}, \\ T'_{21}(u) &= \sum_{i=1}^{p_2} t_{21}^{(i)} u^{p_2-i}, & T'_{12}(u) &= \sum_{i=1}^{p_2} t_{12}^{(i)} u^{p_2-i}. \end{aligned}$$

Let $D(u)' = T'_{21}(u+1)T'_{12}(u)$. Let $y_1, \dots, y_{p_1+p_2}$ be the graded images of the coefficients of $D(u)'$ in $Y_{p_2}(\mathfrak{gl}_2)/(T'_{11}(u), T'_{22}(u))$. The Yangian $Y_{p_2}(\mathfrak{gl}_2)$ is free over its Gelfand-Tsetlin subalgebra generated by $T'_{11}(u)'$ and $T'_{11}(u+1)T'_{22}(u)' - D(u)'$ by [FMO1, Theorem 3.4]. Since the sequence $y_1, \dots, y_{p_1+p_2}$ is obtained from a regular sequence in $\bar{Y}_{p_2}(\mathfrak{gl}_2)$ by substituting zeros instead of some generators, then $y_1, \dots, y_{p_1+p_2}$ is regular by Lemma 8.1. Hence its subsequence $y_{p_2+1}, \dots, y_{p_1+p_2}$ is also regular. Thus the variety $V(y_{p_2+1}, \dots, y_{p_1+p_2}) \subset \mathbb{k}^{2p_2}$ is equidimensional. Now project this variety on the subspace \mathbb{k}^{2p_1} by substituting zeros instead of $t_{12}^i, i = 1, \dots, s$ and $t_{21}^i, i = p_1 + 1, \dots, p_2$. The resulting variety is again equidimensional of pure dimension p_1 . Moreover, this variety coincides with the variety $V(z_{p_2+1}, \dots, z_{p_1+p_2})$ and therefore the sequence $z_{p_2+1}, \dots, z_{p_1+p_2}$ is regular.

Hence we proved

Theorem 8.2. *$W(\pi)$ is free as a right (left) module over the Gelfand-Tsetlin subalgebra.*

Consider the following analog of the *Kostant-Wallach map* ([KoW])

$$KW : \text{Specm } \bar{W}(\pi) \simeq \mathbb{k}^{3p_1+p_2} \rightarrow \text{Specm } \bar{\Gamma} \simeq \mathbb{k}^{2p_1+p_2}.$$

In particular we showed

Corollary 8.3. *The map KW is surjective and the variety $KW^{-1}(0)$ is equidimensional of pure dimension p_1 .*

We also get a good estimate of the size of the fiber for any $\mathbf{m} \in \text{Specm}$.

Theorem 8.4. *Let (p_1, p_2) are the rows of π . For any $\mathbf{m} \in \text{Specm } \Gamma$ the fiber of \mathbf{m} consists of at most $p_1!$ isomorphism classes of irreducible Gelfand-Tsetlin $W(\pi)$ -modules. Moreover, the dimension of the subspace of \mathbf{m} -nilpotents in any such module is bounded by $p_1!$.*

Proof. Since $W(\pi)$ is free over Γ and Γ is a polynomial ring then all conditions of [FO2, Theorem 5.3,(iii)] are satisfied, and hence we have that the fiber of \mathbf{m} consists of at most $p_1!(p_1 + p_2)!$ isomorphism classes of irreducible Gelfand-Tsetlin $W(\pi)$ -modules. But this bound can be improved following [FO2, Corollary 6.1,(2)]. Let $\bar{\Gamma}$ be the integral closure of Γ in L . If $\ell \in \text{Specm } \bar{\Gamma}$ projects to $\mathbf{m} \in \text{Specm } \Gamma$ then we write $\ell = \ell_{\mathbf{m}}$. Note that given $\mathbf{m} \in \text{Specm } \Gamma$ the number of different $\ell_{\mathbf{m}}$ is finite. Moreover, for any $\mathbf{m} \in \text{Specm } \Gamma$ and some fixed $\ell_{\mathbf{m}}$ there exists at most $p_1!$ elements $s \in \mathbb{Z}^{p_1}$ such that $\ell_{\mathbf{m}}$ and $\ell_{\mathbf{m}} + s$ differ by the action of $G = S_{p_1} \times S_{p_1+p_2}$. It immediately implies the statement about the bound for the fiber. The same number bounds the dimension of the subspace of \mathbf{m} -nilpotents by [FO2, Corollary 6.1,(1)]. \square

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REFERENCES

- [A] Arakawa T., *Representation theory of W -algebras*, [arXiv:math/0506056](#).
- [AOV1] Alev J., Ooms A., Van den Bergh M., *The Gelfand-Kirillov conjecture for Lie algebras of dimension at most eight*, J. Algebra 227 (2000), 549-581. Corrigendum, J. Algebra 230 (2000), 749.
- [AOV2] Alev J., Ooms A., Van den Bergh M., *A class of counterexamples to the Gelfand-Kirillov conjecture*, Trans. Amer. Math. Soc. 348 (1996), 1709-1716.
- [BGR] Borho W., Gabriel P., Rentschler R., *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Lecture Notes in Math. vol 357, Springer, Berlin and New York, 1973.
- [BB] Brown J. and Brundan J., *Elementary invariants for centralizers of nilpotent matrices*, J. Austral. Math. Soc., to appear; [arXiv:math/0611024](#).
- [BG] Brown K.A., Goodearl K.R., *Lectures on algebraic quantum groups*, Advance course in Math. CRM Barcelona, vol 2., Birkhauser Verlag, Basel, 2002.
- [BK1] Brundan J. and Kleshchev A., *Shifted Yangians and finite W -algebras*, Adv. Math. **200** (2006), 136-195.
- [BK2] Brundan J. and Kleshchev A., *Representations of shifted Yangians and finite W -algebras*, Memoirs AMS, to appear; [arXiv:math/0508003](#).
- [DFO1] Drozd Yu.A., Ovsienko S.A., Futorny V.M. *On Gelfand-Zetlin modules*, Suppl. Rend. Circ. Mat. Palermo, **26** (1991), 143-147.

- [DFO2] Drozd Yu.A., Ovsienko S.A., Futorny V.M., *Harish-Chandra subalgebras and Gelfand-Zetlin modules*, in: "Finite dimensional algebras and related topics", NATO ASI Ser. C., Math. and Phys. Sci., **424**, (1994), 79-93.
- [EK] Elashvili A.G., Kac V.G., *Classification of good gradings of simple Lie algebras*, Amer. Math. Soc. Transl., Ser.2, 213 (2005), 85-104.
- [Fa] Faith C., *Galois subrings of Ore domains are Ore domains*, Bull. AMS, **78** (1972), no.6 1077-1080.
- [FF] Feigin B., Frenkel E., *Quantization of Drinfeld-Sokolov reduction*, Phys. Lett. B, 246 (1990), 75-81.
- [FMO1] Futorny V., Molev A. and Ovsienko S., *Harish-Chandra modules for Yangians*, Represent. Theory **9** (2005), 426-454.
- [FMO2] Futorny V., Molev A. and Ovsienko S., *Gelfand-Tsetlin bases for representations of finite W -algebras and shifted Yangians*, In: Lie theory and its applications in Physics VII, 2008, Varna: Heron Press, 2007, 352-363.
- [FO1] Futorny V., Ovsienko S., *Galois orders*, [arXiv:math/0610069](#).
- [FO2] Futorny V., Ovsienko S., *Fibers of characters in Harish-Chandra categories*, [arXiv:math/0610071](#).
- [FO3] Futorny V., Ovsienko S., *Kostant's theorem for special filtered algebras*, Bull. London Math. Soc. 37 (2005), 1-13.
- [GG] Gan W.L., Ginzburg V., *Quantization of Slodowy slices*, Int. Math. Res. Notices 5 (2002), 243-255.
- [GK1] Gelfand I.M. et. Kirillov A.A., *Sur les corps liés cor aux algèbres enveloppantes des algèbres de Lie*, Publ. Inst. Hautes Sci., 31, (1966), 5-19.
- [GK2] Gelfand I.M. and Kirillov A.A., *The structure of the Lie field connected with a split semi-simple Lie algebra*, Funktsional. Anal. i Prilozhen. 3 (1969), 6-21.
- [GP] Greuel G-M., Pfister G., *A Singular introduction to commutative algebra*, Springer, 2nd, extended ed., 2008, 690pp.
- [Jo] Joseph A., *Proof of the Gelfand-Kirillov conjecture for solvable Lie algebras*, Proc. Amer. Math. Soc. 45 (1974), 1-10.
- [KW] Kac V., Wakimoto M., *Quantum reduction and representation theory of superconformal algebras*, Adv. Math., 185 (2004), 400-458, Corrigendum, Adv. Math., 193 (2005), 453-455.
- [KRW] Kac V., Roan S.S., Wakimoto M., *Quantum reduction for affine superalgebras*, Comm. Math. Phys., 241 (2003), 307-342.
- [Ko] Kostant B. *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), 101-184.
- [KoW] Kostant B., Wallach N.: *Gelfand-Zeitlin theory from the perspective of classical mechanics I*. In Studies in Lie Theory Dedicated to A. Joseph on his Sixtieth Birthday, Progress in Mathematics, Vol. 243, (2006), 319-364.
- [L] Lynch T.E., *Generalized Whittaker vectors and representation theory*, PhD thesis, MIT, 1979.
- [MCR] McConnell J. C. and Robson J.C., *Noncommutative noetherian rings*, Chichester, Wiley, 1987.
- [Mc] McConnel J.C., *Representations of solvable Lie algebras and the Gelfand-Kirillov conjecture*, Proc. London Math. Soc., 29 (1974), 453-484.
- [M] Molev A., *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI, 2007.
- [Ng] Nghiem X.H., *Reduction de produit semi-directs et conjecture de Gelfand et Kirillov*, Bull. Soc. Math. France (1979), 241-267.

- [O] Ovsienko S., *Finiteness statements for Gelfand–Tsetlin modules*, in: “Algebraic Structures and Their Applications”, Math. Inst., Kiev, 2002.
- [P] Premet A., *Special transverse slices and their enveloping algebras*, Advances Math. 170 (2002), 1-55.
- [RS] Ragoucy E., Sorba P., *Yangian realizations from finite W -algebras*, Comm. Math. Phys., 203 (1999), 551-576.
- [SK] De Sole A., Kac A., *Finite vs affine W -algebras*, Japanese J. Math., 1 (2006), 137-261.

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